# The Graph of a Rational Function

## Analyzing the Graph of a Rational Function

1. Factor the numerator and denominator of the rational function and find the **domain** by setting the denominator not equal to zero. (Do this before simplifying).
2. Simplify the rational function.
3. Find any holes in the graph. Set any factors that appeared in both the numerator and denominator of the unsimplified function equal to zero to find the $x$-coordinate of the hole. Plug the $x$-coordinate into the simplified function to find the $y$-coordinate of the hole.
4. Find the ***x*-intercepts** by setting the numerator of the simplified function equal to zero. Find the ***y*-intercept** by plugging in zero for $x$.
5. Find the **vertical asymptotes** by setting the denominator of the simplified rational function equal to zero.
6. Find the **horizontal asymptotes or oblique asymptotes:**
   a) If the ***degree of the numerator < the degree of the denominator***, then the horizontal asymptote is $y = 0$.
   b) If the ***degree of the numerator = the degree of the denominator***, then the horizontal asymptote is $y = \dfrac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$.
   c) If the ***degree of the numerator > the degree of the denominator***, then the graph has an oblique asymptote (if the degree of the numerator is one more than the degree of the denominator), or the graph approaches a quadratic or higher degree function (if the degree of the numerator is more than one higher than the degree of the denominator). To determine the end behavior, use long division.
7. Use the $x$-intercepts and vertical asymptotes as **critical points** to divide the graph into intervals and determine where it is above the $x$-axis ($y$ is positive) and where it is below the $x$-axis ($y$ is negative). Do this by choosing a number $x$ in each interval and evaluating $R$ there.
8. Analyze the behavior of the graph near each asymptote.
9. Put together all of the information to graph the function.

**Example:** Analyze the Graph of the Rational Function $R(x) = \dfrac{x+2}{x^2-9} = \dfrac{x+2}{(x+3)(x-3)}$

Domain: $(x+3)(x-3) \neq 0$

$\boxed{\{x \mid x \neq -3, 3\}}$

y-int: $R(0) = \dfrac{0+2}{0^2-9} = -\dfrac{2}{9}$ $\boxed{(0, -\tfrac{2}{9})}$

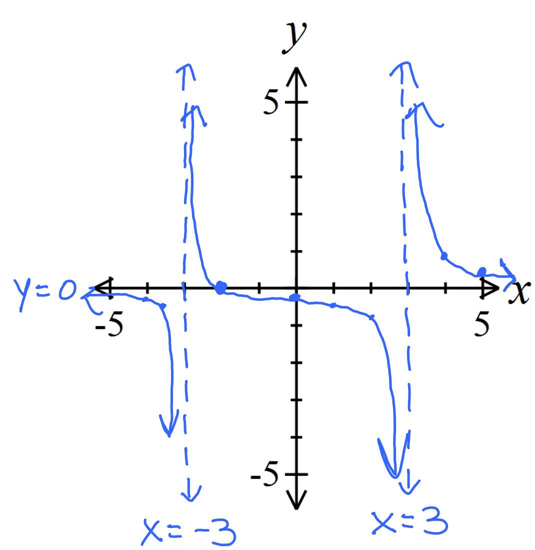
x-int: $x+2=0$ $x=-2$ $\boxed{(-2, 0)}$

Vert. Asymptote: $(x+3)(x-3)=0$

$\boxed{x=-3,\ x=3}$

Horiz. Asymptote: $\boxed{y=0}$

(Degree num < Degree denom)




| x | y |
|---|---|
| -4 | -.3 |
| -3.5 | -.5 |
| -1 | -.1 |
| 1 | -.4 |
| 2 | -.8 |
| 4 | .9 |
| 5 | .4 |

**Example:** Analyze the Graph of the Rational Function $R(x)=\dfrac{x^3+1}{x^2+2x}=\dfrac{(x+1)(x^2-x+1)}{x(x+2)}$

Domain: $x(x+2)\neq 0$

$\boxed{\{x \mid x\neq 0,-2\}}$

y-int: $\boxed{\text{no y-int}}$ (undefined @ $x=0$)

x-int: $(x+1)(x^2-x+1)=0$

$x=-1$ $\boxed{(-1,0)}$

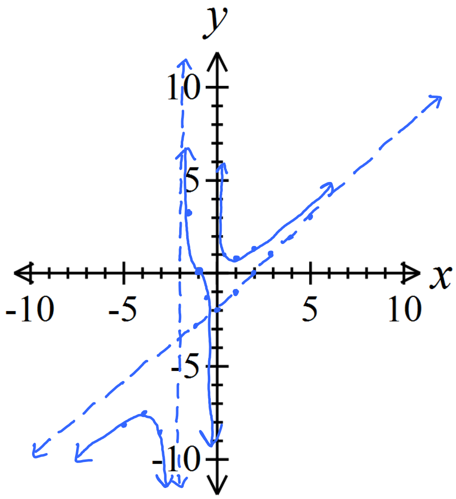
vertical asymptotes:

$x(x+2)=0$

$\boxed{x=-2,\ x=0}$

oblique asymptote: $\boxed{y=x-2}$

$$\begin{array}{r} x-2 \\ x^2+2x\overline{)\,x^3+0x^2+0x+1} \\ -(x^3+2x^2) \\ \hline -2x^2+0x \\ -(-2x^2-4x) \\ \hline 4x+1 \end{array}$$



| x | y |
|---|---|
| -5 | -8.3 |
| -4 | -7.9 |
| -3 | -8.7 |
| -1.5 | 3.2 |
| -.5 | -1.2 |
| .5 | .9 |
| 1 | .7 |
| 2 | 1.1 |

**Example:** Analyze the Graph of the Rational Function $R(x)=\dfrac{x^4+x^2+1}{x^2-1}=\dfrac{x^4+x^2+1}{(x+1)(x-1)}$

Domain: $(x+1)(x-1)\neq 0$

$\boxed{\{x \mid x\neq -1,\ x\neq 1\}}$

y-int: $R(0)=\dfrac{0^4+0^2+1}{0^2-1}=-1$ $\boxed{(0,-1)}$

x-ints: $x^4+x^2+1=0$ $u=x^2$

$u^2+u+1=0$

$u=\dfrac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}=\dfrac{-1\pm\sqrt{-3}}{2}$ ← imaginary

$\boxed{\text{no x-ints}}$

vertical asymptotes:

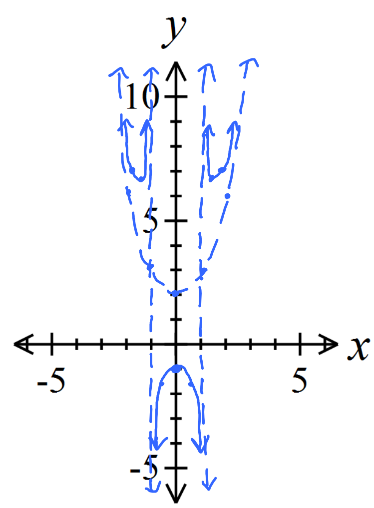
$(x+1)(x-1)=0$

$\boxed{x=-1,\ x=1}$

no horizontal/oblique asymptotes:

$$\begin{array}{r} x^2+2 \\ x^2-1\overline{)\,x^4+x^2+1} \\ -(x^4-x^2) \\ \hline 2x^2+1 \\ -(2x^2-2) \\ \hline 3 \end{array}$$

$\boxed{\text{y=x²+2 acts like an asymptote}}$



| x | y |
|---|---|
| -3 | 11.4 |
| -2 | 7 |
| -1.5 | 6.65 |
| -.5 | -1.75 |
| 0 | -1 |
| .5 | -1.75 |
| 1.5 | 6.65 |
| 2 | 7 |
| 3 | 11.4 |

**Example:** Analyze the Graph of the Rational Function $R(x)=\dfrac{x^2+x-12}{x^2-4}=\dfrac{(x+4)(x-3)}{(x+2)(x-2)}$

Domain: $(x+2)(x-2)\neq 0$

$\boxed{\{x \mid x\neq -2, 2\}}$

y-int: $R(0)=\dfrac{0^2+0-12}{0^2-4}=3$ $\boxed{(0,3)}$

x-int: $(x+4)(x-3)=0$ $\boxed{(-4,0) \text{ \& } (3,0)}$

$x=-4, 3$

vertical asymptotes:

$(x+2)(x-2)=0$

$\boxed{x=-2,\ x=2}$

horizontal asymptote:

$y=\frac{1}{1}$ $\boxed{y=1}$

(deg num = deg denom)

crosses horiz asymp?

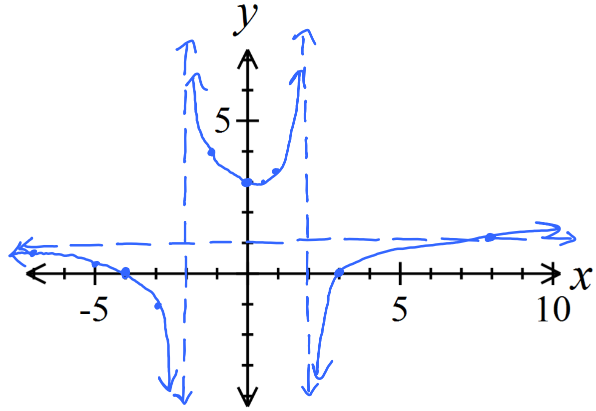
$\dfrac{x^2+x-12}{x^2-4}=1$

$x^2+x-12=x^2-4$

$x-12=-4$

$x=8$

$\boxed{\text{crosses horizontal asymptote @ } (8,1)}$




| | |
|---|---|
| -7 | .7 |
| -5 | .4 |
| -3 | -1.2 |
| -1 | 4 |
| 1 | 3.3 |

**Example:** Analyze the Graph of the Rational Function $R(x)=\dfrac{x^2+3x-10}{x^2+8x+15}=\dfrac{(x+5)(x-2)}{(x+5)(x+3)}=\dfrac{x-2}{x+3}$

Domain: $(x+5)(x+3)\neq 0$

$\boxed{\{x \mid x\neq -5, -3\}}$

hole vert. asymp

y-int: $R(0)=\dfrac{0-2}{0+3}=-\dfrac{2}{3}$ $\boxed{(0,-\frac{2}{3})}$

x-int: $x-2=0$ $x=2$ $\boxed{(2,0)}$

vertical asymptotes:

$x+3=0$ $\boxed{x=-3}$

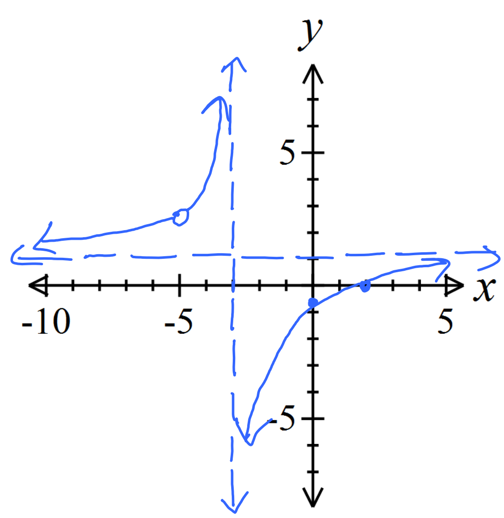
horizontal asymptote: $\boxed{y=1}$

$y=\frac{1}{1}$

(deg. numer = deg. denom)

hole: $x=-5$

$y=\dfrac{-5-2}{-5+3}=\dfrac{-7}{-2}=\dfrac{7}{2}$ $\boxed{(-5,\frac{7}{2})}$




Crosses horiz. asymptote? $\boxed{\text{no}}$

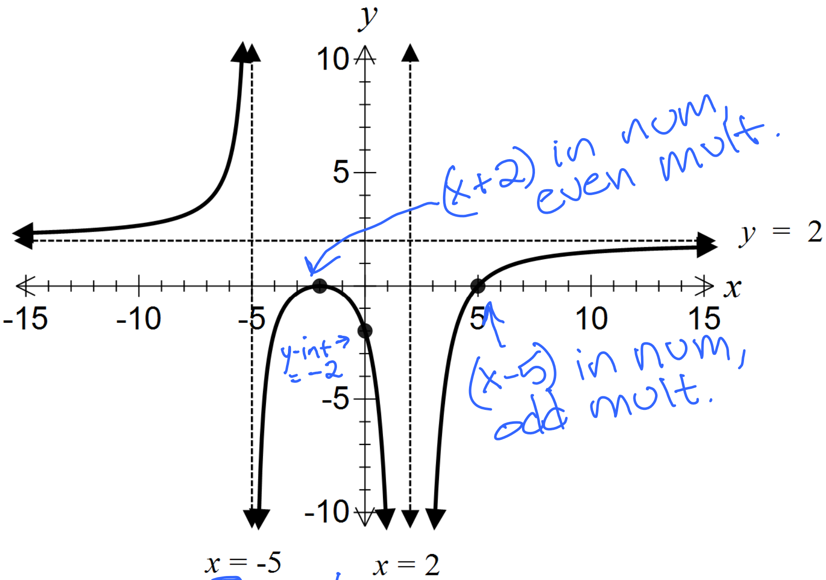
$\dfrac{x-2}{x+3}=1$

$x-2=x+3$

$-2\neq 3$

**Example:** Find a rational function that might have the graph shown below.

★ **Note:** If the graph goes the same direction on both sides of an asymptote (approaches $\infty$ on both sides or approaches $-\infty$ on both sides), the related factor in the denominator has an even multiplicity. If the graph goes in opposite directions on the two sides of an asymptote (approaches $\infty$ on one side and $-\infty$ on the other), the related factor in the denominator has an odd multiplicity.




(x+2) in num, even mult.

y-int = −2

(x−5) in num, odd mult.

Deg numer = Deg denom

Ratio of leading coefficients is 2

(x+5) in denom, odd multiplicity

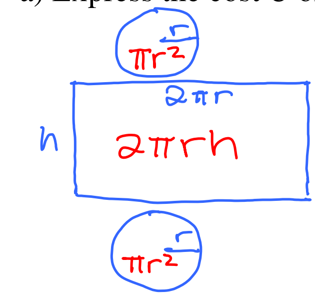
(x−2) in denom, even multiplicity

$$\boxed{R(x) = \frac{2(x+2)^2(x-5)}{(x+5)(x-2)^2}}$$

**Example:** Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a right circular cylinder with a capacity of 500 cubic centimeters. The top and bottom of the can are made of special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the cans are made of material that costs 0.02¢ per square centimeter.

a) Express the cost $C$ of material for the can as a function of the radius $r$ of the can.




Area of top/bottom: $2\pi r^2$

Area of sides: $2\pi r h$

$C = (2\pi r^2)(.05) + (2\pi r h)(.02)$

$C = .1\pi r^2 + .04\pi r h$

$V = \pi r^2 h$

$500 = \pi r^2 h$

$h = \dfrac{500}{\pi r^2}$

$C(r) = .1\pi r^2 + .04\pi r\left(\dfrac{500}{\pi r^2}\right)$

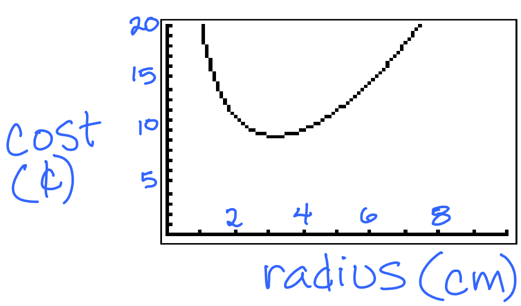
$\boxed{C(r) = .1\pi r^2 + \dfrac{20}{r} = \dfrac{.1\pi r^3 + 20}{r}}$

b) Find any vertical asymptotes. Discuss the cost of the can near any vertical asymptotes.

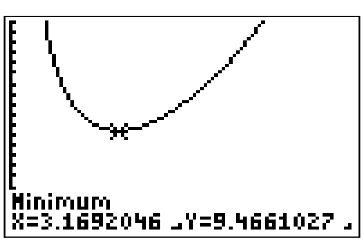
$r = 0$ is a vertical asymptote.

As $r$ gets close to zero, the cost increases.

c) Use a graphing calculator to graph the function $C = C(r)$.




d) What value of $r$ will result in the least cost? What is the least cost?




The cost is lowest when the radius is 3.17 cm

The least cost is 9.47¢ per can